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## SUMMARY

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JOHN M. ANDREW, Col, USAF  
Chair, Basic Sciences Division  
Permanent Professor and Head  
Department of Mathematical Sciences

1 Tab  
1. Journal article

# Constructing $2 \times 2$ Bricks from Unitary Numerical Semigroups

Trae Holcomb  
trae.holcomb@usafa.edu

Department of Mathematical Sciences  
2354 Fairchild Dr Suite 6D124  
United States Air Force Academy  
USAF Academy, CO 80840-6252  
Fax: 719-333-2114

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**Abstract.** *This paper investigates numerical semigroups that yield  $2 \times 2$  bricks. We demonstrate the existence of an infinite family of  $2 \times 2$  bricks that includes all of the perfect  $2 \times 2$  bricks. We provide a formula for the Frobenius numbers of these semigroups as well as a necessary and sufficient condition for the semigroups to be symmetric.*

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## 1. Introduction

The motivation for studying unitary numerical semigroups and bricks began with the study of torsion in tensor products of modules over one-dimensional local domains. Much of the discussion that follows about the relationship between these topics comes from [6].

Let  $(R, \mathfrak{m})$  be a monomial domain with monomial fractional ideal  $I$ . Let  $K$  be the quotient field of  $R$  and let  $\nu : K^* \rightarrow \mathbf{Z}$  be the standard valuation mapping. Then  $S = \nu(R)$  is a numerical semigroup. For a relative ideal  $J$  of  $S$ , we use the notation  $\mu_S(J)$  to represent the number of elements in the minimal generating set for  $J$ . In this case,

- (1)  $\mu_S(S \setminus \{0\}) = \text{embeddim}(R)$ ,
- (2)  $\nu(I)$  is a relative ideal of  $S$  and  $\mu_S(\nu(I)) = \mu_R(I)$ ,
- (3)  $\nu(I^{-1}) = S - \nu(I)$ , and
- (4)  $\nu(II^{-1}) = \nu(I) + (S - \nu(I))$ .

We know that  $I \otimes_R I^{-1}$  has non-zero torsion whenever  $\mu_R(I) + \mu_R(I^{-1}) > \mu_R(II^{-1})$ . Therefore, if we hope to find an  $I$  such that  $I \otimes_R I^{-1}$  is torsion free, we must have  $\mu_R(I) + \mu_R(I^{-1}) = \mu_R(II^{-1})$ . In this case,

$$\mu_S(\nu(I)) + \mu_S(S - \nu(I)) = \mu_S(\nu(II^{-1})).$$

This equality motivated the definition and study of semigroup bricks. In fact, a  $k \times m$  brick was defined in [9] so that the above equality would hold. That is,  $(S, \nu(I))$  is a  $k \times m$  brick where  $k = \mu_R(I)$  and  $m = \mu_R(I^{-1})$  precisely when  $\mu_S(\nu(I)) + \mu_S(S - \nu(I)) = \mu_S(\nu(II^{-1}))$ . The investigation in [9] also introduced the notion of perfect bricks as well as balanced and unitary numerical semigroups having minimal generating sets of four elements. The main result showed that every unitary numerical semigroup yields a perfect  $2 \times 2$  brick. The authors in [10] proved the converse, thereby showing that unitary numerical semigroups and perfect  $2 \times 2$  bricks are equivalent ideas. These two investigations completely classified the infinite family of perfect  $2 \times 2$  bricks.

However, the perfect  $2 \times 2$  bricks form only a small subset of the set of all  $2 \times 2$  bricks. Furthermore, we know from [13] that  $R$  is Gorenstein when  $S$  is symmetric. By (3.2) in [7], this means that if  $(S, \nu(I))$  is a symmetric  $2 \times 2$  brick, then  $I \otimes_R I^{-1}$  has non-zero torsion. Since all of the perfect  $2 \times 2$  bricks are symmetric, we know  $I \otimes_R I^{-1}$  has non-zero torsion whenever  $(S, \nu(I))$  is a perfect  $k \times m$  brick. Identifying and classifying all  $2 \times 2$  bricks

is therefore important for identifying contexts in which  $I \otimes_R I^{-1}$  might be torsion free.

The authors in [9] conjectured that there is a natural relationship between  $2 \times 2$  bricks and perfect  $2 \times 2$  bricks, that is, that there is a natural relationship between  $2 \times 2$  bricks and unitary numerical semigroups. In this paper, we discuss this relationship and construct (in Theorem 3.2) a family of  $2 \times 2$  bricks for each perfect  $2 \times 2$  brick, thereby creating an infinite collection of candidates satisfying the condition  $\mu_S(\nu(I)) + \mu_S(S - \nu(I)) = \mu_S(\nu(II^{-1}))$ .

As mentioned above, knowing when  $2 \times 2$  bricks are not symmetric is particularly valuable for identifying contexts where  $I \otimes_R I^{-1}$  could be torsion free. In Corollary 3.12, we provide a necessary and sufficient condition for the numerical semigroups defined by Theorem 3.2 to be symmetric as well as a formula for the Frobenius number of each semigroup.

The interested reader can find additional details concerning the investigation of torsion in tensor products in [1], [3], [11] and [12]. Many results pertaining to the key equality above can be found in [6], [7], and [8]. Suggested background reading on the connections between numerical semigroups and commutative algebra include [2], [5] and [13].

## 2. Preliminaries

The definitions and notation pertinent to this investigation are given below. Although many are similar to those in [9], we repeat them here for convenience.

### Definitions and Notation 2.1.

(a) A *numerical semigroup*  $S$  is a subset of the non-negative integers  $\mathbf{N}$  that contains 0, is closed under addition, and such that  $\mathbf{N} \setminus S$  is finite. If  $G$  is the smallest subset of  $S$  such that every non-zero element of  $S$  is a sum of elements from  $G$ , then we say  $G$  is the *minimal generating set* of  $S$  and we write  $S = \langle G \rangle$ .

(b) The *multiplicity* of  $S$ , denoted  $e(S)$ , is the smallest positive element of  $S$ . If  $S = \langle a_1, \dots, a_k \rangle$  where  $0 < a_1 < \dots < a_k$ , then  $e(S) = a_1$ .

(c) The *Frobenius number* of  $S$ , denoted  $g(S)$ , is the largest element in  $\mathbf{Z} \setminus S$  and the *conductor* of  $S$  is the smallest integer  $x$  such that  $x + m \in S$  for all  $m \in \mathbf{N}$ . That is, the conductor of  $S$  is  $g(S) + 1$ .



(d) We say that  $S$  is *symmetric* if for every  $x \in \mathbf{Z}$  we have  $x \in S$  if and only if  $g(S) - x \notin S$ .

(e) Let  $m$  be a non-zero element of  $S$ . We define the *Apéry set of  $S$  with respect to  $m$*  by  $Ap(S, m) = \{s \in S \mid s - m \notin S\}$ .

(f) A *relative ideal*  $I$  of  $S$  is a set of integers such that  $I + S \subseteq I$  and  $s + I \subset S$  for some  $s \in S$ . It follows that  $I$  can be represented as a finite union of cosets  $z + S$  where  $z \in \mathbf{Z}$ . The notation  $I = (z_1, \dots, z_k)$  means  $I = (z_1 + S) \cup \dots \cup (z_k + S)$  and  $I$  cannot be written as a union of cosets for any proper subset of  $\{z_1, \dots, z_k\}$ . We refer to  $\{z_1, \dots, z_k\}$  as *the minimal generating set of  $I$* . If we refer to an element  $x$  of  $I$  being a *generator for  $I$* , we will mean that  $x$  is contained in the minimal generating set of  $I$ . We recognize that this is a slight abuse of terminology, but it will ease the communication of some of the results.

(g) If  $I$  and  $J$  are relative ideals of  $S$ , we define their sum by  $I + J = \{i + j \mid i \in I, j \in J\}$ .

(h)  $\mu_S(I)$  represents the number of elements in the minimal generating set for  $I$  as a relative ideal of  $S$ . We say a relative ideal  $I$  is *principal* if  $\mu_S(I) = 1$  and *non-principal* if  $\mu_S(I) \geq 2$ .

(i) The *dual* of  $I$  in  $S$  is  $S - I = \{z \in \mathbf{Z} \mid z + I \subseteq S\}$ . Note that  $S - I$  is also a relative ideal of  $S$ . It is clear from the definition of  $S - I$  that  $I + (S - I) \subseteq S \setminus \{0\}$  whenever  $I$  is non-principal.

(j) A set of positive integers  $B = \{a_1, a_2, a_3, a_4\}$  such that  $a_1 < a_2 < a_3 < a_4$  is said to be *balanced* provided that  $a_1 + a_4 = a_2 + a_3$ . A numerical semigroup  $S = \langle a_1, a_2, a_3, a_4 \rangle$  is said to be *balanced* provided that  $\{a_1, a_2, a_3, a_4\}$  is a balanced set. Let  $D = \gcd(a_1, a_4)$  and  $E = \gcd(a_2, a_3)$ . Note that for  $S$  to be a numerical semigroup,  $D$  and  $E$  must be relatively prime. We will write  $a_1 = q_1 D, a_2 = q_2 E, a_3 = q_3 E, a_4 = q_4 D$ . The quantity  $a_1 + a_4 = a_2 + a_3$  is called the *common sum* of  $S$  and is denoted by  $CS(S)$ . The quantity  $\frac{CS(S)}{DE}$  is called the *common quotient* of  $S$  and is denoted by  $CQ(S)$ . We say  $S$  is *unitary* provided  $CQ(S) = 1$ . It was observed in [9] (1.7) that  $S$  is unitary if and only if  $q_2 + q_3 = D$  if and only if  $q_1 + q_4 = E$  if and only if  $CS(S) = DE$ . Note that we will call a set *unitary* if it is the minimal generating set of a unitary numerical semigroup.

(k) Let  $S$  be a numerical semigroup and let  $I$  be a non-principal relative ideal of  $S$ . We refer to the pair  $(S, I)$  as a  $k \times m$  brick provided  $\mu_S(I) = k$ ,  $\mu_S(S - I) = m$ , and  $\mu_S(I + (S - I)) = \mu_S(I) \mu_S(S - I) = km$ . If  $(S, I)$  is a  $k \times m$  brick and  $I + (S - I) = S \setminus \{0\}$ , then we say  $(S, I)$  is a *perfect*  $k \times m$  brick.

**Standing Assumption 2.2.** If  $(S, I)$  is a  $2 \times 2$  brick, then we may assume  $I = (0, n)$  where  $0 < n \notin S$ .

**Proof.** See (2.5) in [6].

**Lemma 2.3.** If  $(S, I)$  is a  $2 \times 2$  brick, then the minimal generating set for  $I + (S - I)$  is balanced.

**Proof.** Suppose  $(S, I)$  is a  $2 \times 2$  brick. Then as above, we may assume  $I = (0, n)$  where  $0 < n \notin S$  and hence  $S - I = (b_1, b_2)$  where  $b_1, b_2 \in S$ ,  $0 < b_1 < b_2$  and  $b_2 - b_1 \notin S$ . Thus,  $I + (S - I) = (b_1, b_1 + n, b_2, b_2 + n)$  where either  $b_1 < b_1 + n < b_2 < b_2 + n$  or  $b_1 < b_2 < b_1 + n < b_2 + n$ . In either case, the minimal generating set for  $I + (S - I)$  is balanced.

**Standing Assumption 2.4.** Suppose that  $(S, I)$  is a  $2 \times 2$  brick with  $I = (0, n)$  and  $I + (S - I) = (a_1, a_2, a_3, a_4)$ . From the proof of Lemma 2.3, it is either the case that  $n = a_2 - a_1 = a_4 - a_3$  so that  $S - I = (a_1, a_3)$  or  $n = a_3 - a_1 = a_4 - a_2$  so that  $S - I = (a_1, a_2)$ . For the remainder of this investigation, we will assume that  $n = a_2 - a_1 = a_4 - a_3$ . The statements and proofs for all results are valid in the case  $n = a_3 - a_1 = a_4 - a_2$  by reversing the subscripts 2 and 3 as appropriate.

**Definitions and Notation 2.5.** Let  $(S, I)$  be a  $2 \times 2$  brick, where  $I + (S - I) = (a_1, a_2, a_3, a_4)$ ,  $I = (0, n = a_2 - a_1)$ , and  $S - I = (a_1, a_3)$ . Let  $k = \gcd(a_1, a_2, a_3, a_4)$ . We know that  $\{a_1/k, a_2/k, a_3/k, a_4/k\}$  is the minimal generating set of a numerical semigroup which we will call the *underlying semigroup* of  $(S, I)$  and denote by  $S_I$ . In the event that  $k = 1$ , then  $S_I = \langle a_1, a_2, a_3, a_4 \rangle$ .

**Example 2.6.** Let  $S = \langle 14, 21, 27, 36, 45 \rangle$ . We immediately see that  $e(S) = 14$ , and it is easy to verify that  $g(S) = 88$ . Let  $I = (0, 1) = (0 + S) \cup (1 + S)$  be a relative ideal of  $S$ . It is straightforward to determine that  $S - I = (27, 35)$ . By adding each of the generators of  $I$  to each of the generators of  $S - I$ , we find that  $\{27, 28, 35, 36\}$  is a generating set for

$I + (S - I)$ . A quick check reveals that this generating set is minimal. That is,  $I + (S - I) = (27, 28, 35, 36)$ . Therefore,  $(S, I)$  is a  $2 \times 2$  brick. Since  $\gcd(27, 28, 35, 36) = 1$ ,  $S_I = \langle 27, 28, 35, 36 \rangle$ . We note that  $S_I$  is unitary with  $D = 9$  and  $E = 7$ . From (2.1) in [9], we conclude that  $(S_I, J)$  is a perfect  $2 \times 2$  brick, where  $J = (0, 1)$ .

### 3. An Infinite Family of $2 \times 2$ Bricks

In [9], the authors conjectured that for any  $2 \times 2$  brick,  $(S, I)$ , the generating set for  $I + (S - I)$  must be unitary. The following example demonstrates that this is not the case.

**Example 3.1.** Let  $S = \langle 15, 20, 25, 28, 42 \rangle$ , and let  $I = (0, 2)$ . It is easy to verify that  $S - I = (28, 40)$  and  $I + (S - I) = (28, 30, 40, 42)$ . Therefore,  $(S, I)$  is a  $2 \times 2$  brick, but  $\{28, 30, 40, 42\}$  is not unitary since its common quotient is  $\frac{28+42}{14+10} = \frac{1}{2}$ . However, we notice that  $S_I = \langle 14, 15, 20, 21 \rangle$ , which is unitary with  $D = 7$  and  $E = 5$ .

Although the minimal generating set for  $I + (S - I)$  in the above example is not unitary, we notice that it is a constant multiple of a unitary set. We believe this to be true for every  $2 \times 2$  brick.

In [9], the authors constructed an infinite family of perfect  $2 \times 2$  bricks that was subsequently shown in [10] to be the entire collection of perfect  $2 \times 2$  bricks. Here, we take a step toward characterizing all  $2 \times 2$  bricks in a similar manner.

Theorem 3.2 below is the main result of this investigation. It constructs an infinite family of  $2 \times 2$  bricks that includes all of the perfect  $2 \times 2$  bricks. We believe this to be the entire collection of  $2 \times 2$  bricks. However, proving this appears to be a significant effort in its own right and will therefore be left for future work.

**Theorem 3.2.** Let  $S = \langle a_1 = q_1D, a_2 = q_2E, a_3 = q_3E, a_4 = q_4D \rangle$  be a unitary numerical semigroup with  $D = \gcd(a_1, a_4)$ ,  $E = \gcd(a_2, a_3)$ , and  $n = q_2E - q_1D = q_4D - q_3E$ . Let  $q_D$  and  $q_E$  be positive integers such that  $\gcd(q_D D, q_E E) = 1$ , and let  $T_D = \langle s_1, s_2, \dots, s_m \rangle$  and  $T_E = \langle r_1, r_2, \dots, r_k \rangle$  be numerical semigroups satisfying the conditions

- (1)  $q_1, q_4 \in T_D$ , but  $q_E(q_4 - q_1) \notin T_D$ , and
- (2)  $q_2, q_3 \in T_E$ , but  $q_D(q_3 - q_2) \notin T_E$ .



Let  $G_D = \{(q_D D)s_1, (q_D D)s_2, \dots, (q_D D)s_m\}$  and  $G_E = \{(q_E E)r_1, (q_E E)r_2, \dots, (q_E E)r_k\}$ . Let  $T$  be the numerical semigroup generated by  $G_D \cup G_E$ . Then

$$T = \langle (q_D D)s_1, (q_D D)s_2, \dots, (q_D D)s_m, (q_E E)r_1, (q_E E)r_2, \dots, (q_E E)r_k \rangle;$$

that is, the minimal generating set of  $T$  is  $G_D \cup G_E$ . Furthermore,  $(T, I)$  is a  $2 \times 2$  brick, where  $I = (0, q_D q_E n)$  and  $T - I = (q_D q_E q_1 D, q_D q_E q_3 E)$ . That is, the minimal generating set of  $I + (T - I)$  is  $q_D q_E$  times the minimal generating set of the unitary numerical semigroup  $S$  and hence  $T_I = S$ .

Before proving Theorem 3.2, we lay some groundwork and illustrate it with an example.

**Definition 3.3.** Let  $S_1$  and  $S_2$  be numerical semigroups minimally generated by  $\{n_1, \dots, n_r\}$  and  $\{n_{r+1}, \dots, n_e\}$ , respectively. Let  $\lambda \in S_1 \setminus \{n_1, \dots, n_r\}$  and  $\mu \in S_2 \setminus \{n_{r+1}, \dots, n_e\}$  such that  $\gcd(\lambda, \mu) = 1$ . By Lemma 9.8 in [14],

$$\{\mu n_1, \dots, \mu n_r, \lambda n_{r+1}, \dots, \lambda n_e\}$$

is the minimal generating set of a numerical semigroup,  $S$ , and we say that  $S$  is a *gluing* of  $S_1$  and  $S_2$ .

**Notation and Observations 3.4.** Based on the form of the generating set for  $T$  in Theorem 3.2, we note that if  $x \in T$ , then  $x \in (q_D D)T_D + (q_E E)T_E$ .

Since  $q_1, q_4 \in T_D$  and  $E = q_1 + q_4$ ,  $E \in T_D$  and  $E$  is not in the minimal generating set of  $T_D$ . It follows that  $q_E E$  is in  $T_D$  and not in the minimal generating set of  $T_D$ . Since  $D = q_2 + q_3$ , we can similarly see that  $q_D D$  is in  $T_E$  and not in the minimal generating set of  $T_E$ . Since  $\gcd(q_D D, q_E E) = 1$ ,  $T = (q_D D)T_D + (q_E E)T_E$  is a gluing of  $T_D$  and  $T_E$  and

$$T = \langle (q_D D)s_1, (q_D D)s_2, \dots, (q_D D)s_m, (q_E E)r_1, (q_E E)r_2, \dots, (q_E E)r_k \rangle.$$

**Example 3.5.** As we saw in Example 2.6, if  $T = \langle 14, 21, 27, 36, 45 \rangle$  and  $I = (0, 1)$ , then  $(T, I)$  is a  $2 \times 2$  brick with  $T_I = \langle 27, 28, 35, 36 \rangle$ . This construction is of the type described by Theorem 3.2. That is, if we let  $S = \langle 27, 28, 35, 36 \rangle$ , then  $S$  is a unitary numerical semigroup with  $D = 9$ ,  $E = 7$ ,  $q_1 = 3$ ,  $q_2 = 4$ ,  $q_3 = 5$ ,  $q_4 = 4$ , and  $n = 1$ . If we let  $T_D = \langle 3, 4, 5 \rangle$ ,



$T_E = \langle 2, 3 \rangle$ , and  $q_D = q_E = 1$ , then  $q_1, q_4 \in T_D$  and  $q_E(q_4 - q_1) = 1 \notin T_D$ . We also have  $q_2, q_3 \in T_E$  and  $1 = q_D(q_3 - q_2) \notin T_E$ . Since  $T = \langle 14, 21, 27, 36, 45 \rangle = \langle 3q_DD, 4q_DD, 5q_DD, 2q_EE, 3q_EE \rangle$ , Theorem 3.2 guarantees that  $(T, I)$  is a  $2 \times 2$  brick with  $T_I = S$ .

**Example 3.6.** As we saw in Example 3.1, if  $T = \langle 15, 20, 25, 28, 42 \rangle$  and  $I = (0, 2)$ , then  $(T, I)$  is a  $2 \times 2$  brick with  $T_I = \langle 14, 15, 20, 21 \rangle$ . This construction is of the type described by Theorem 3.2. That is, if we let  $S = \langle 14, 15, 20, 21 \rangle$ , then  $S$  is a unitary numerical semigroup with  $D = 7$ ,  $E = 5$ ,  $q_1 = 2$ ,  $q_2 = 3$ ,  $q_3 = 4$ ,  $q_4 = 3$ , and  $n = 1$ . If we let  $T_D = \langle 2, 3 \rangle$ ,  $T_E = \langle 3, 4, 5 \rangle$ ,  $q_D = 2$ , and  $q_E = 1$ , then  $q_1, q_4 \in T_D$  and  $q_E(q_4 - q_1) = 1 \notin T_D$ . We also have  $q_2, q_3 \in T_E$  and  $2 = q_D(q_3 - q_2) \notin T_E$ . Since  $T = \langle 15, 20, 25, 28, 42 \rangle = \langle 3q_EE, 4q_EE, 5q_EE, 2q_DD, 3q_DD \rangle$ , Theorem 3.2 guarantees that  $(T, I)$  is a  $2 \times 2$  brick with  $T_I = S$ .

**Remark 3.7.** It is well-known that  $Ap(S, m)$  consists precisely of the minimal representatives in  $S$  of the congruence classes modulo  $m$ . If  $P$  and  $Q$  are relatively prime positive integers and  $S = \langle P, Q \rangle$ , it follows that  $Ap(S, P) = \{0, Q, \dots, (P-1)Q\}$ . Therefore, if  $q \in \mathbb{N}$  such that  $q < P$ , then  $qQ \notin P + S$ . Similarly, if  $p \in \mathbb{N}$  such that  $p < Q$ , then  $pP \notin Q + S$ .

**Lemma 3.8.** Let  $\{a_1 = q_1D, a_2 = q_2E, a_3 = q_3E, a_4 = q_4D\}$  be a unitary set with  $D = \gcd(a_1, a_4)$ ,  $E = \gcd(a_2, a_3)$ ,  $n = q_2E - q_1D = q_4D - q_3E$ , and  $m = q_3E - q_1D = q_4D - q_2E$ . Let  $q_D$  and  $q_E$  be positive integers such that  $\gcd(q_DD, q_EE) = 1$ . Then neither  $q_Dq_En$  nor  $q_Dq_Em$  are in  $\langle q_DD, q_EE \rangle$ .

**Proof.** Assume  $q_Dq_En = c_Dq_DD + c_Eq_EE$  for some  $c_D, c_E \in \mathbb{N}$ . Since  $\gcd(q_DD, q_EE) = 1$ , we have

$$\begin{aligned} c_Dq_DD + c_Eq_EE &= q_Dq_En = q_Dq_E(q_2E - q_1D) \\ \Leftrightarrow (q_Eq_1 + c_D)q_DD &= (q_Dq_2 - c_E)q_EE \\ \Rightarrow (q_Dq_2 - c_E) &= kq_DD \text{ for some } k \in \mathbb{N}. \end{aligned}$$

But,  $0 \leq q_Dq_2 - c_E \leq q_Dq_2 < q_D(q_2 + q_3) = q_DD$  so that  $k = 0$  and  $q_Dq_2 = c_E$ . However,  $c_Dq_DD + c_Eq_EE = q_Dq_E(q_2E - q_1D)$  then implies that  $c_Dq_DD = -q_Eq_1q_DD < 0$ , which is impossible. A similar argument shows that  $q_Dq_Em$  is not in  $\langle q_DD, q_EE \rangle$ .

**Proof of Theorem 3.2.** We saw above that

$$T = \langle (q_DD)s_1, (q_DD)s_2, \dots, (q_DD)s_m, (q_EE)r_1, (q_EE)r_2, \dots, (q_EE)r_k \rangle.$$

Let  $I = (0, q_D q_E n)$ . We need to show that  $T - I = (q_D q_E q_1 D, q_D q_E q_3 E)$  and  $I + (T - I) = (q_D q_E q_1 D, q_D q_E q_2 E, q_D q_E q_3 E, q_D q_E q_4 D)$ . Since  $q_1, q_4 \in T_D$ , we see that  $q_D D q_1, q_D D q_4 \in T$  and hence  $q_D q_E q_1 D, q_D q_E q_4 D \in T$ . We similarly see that  $q_D q_E q_2 E, q_D q_E q_3 E \in T$ . Hence,  $q_D q_E S \subseteq T$ . By (2.1) in [9],  $(S, J)$  is a perfect  $2 \times 2$  brick, where  $J = (0, n)$ . That is,  $S - J = (q_1 D, q_3 E)$ . Since  $q_1 D, q_3 E \in S - J$ ,  $q_1 D + n, q_3 E + n \in S$ . Therefore,  $q_D q_E q_1 D + q_D q_E n, q_D q_E q_3 E + q_D q_E n \in q_D q_E S \subseteq T$  so that  $q_D q_E q_1 D, q_D q_E q_3 E \in T - I$ . By Lemma 3.8,  $q_D q_E (q_3 E - q_1 D) \notin \langle q_D D, q_E E \rangle$  so that  $q_D q_E (q_3 E - q_1 D) \notin T$ . Since  $q_D q_E q_3 E \notin q_D q_E q_1 D + T$ ,  $\{q_D q_E q_1 D, q_D q_E q_3 E\}$  is the minimal generating set of an ideal of  $T$  and  $(q_D q_E q_1 D, q_D q_E q_3 E) \subseteq T - I$ .

Let  $x \in T - I$ . We will show that  $x \in (q_D q_E q_1 D + T) \cup (q_D q_E q_3 E + T)$ , thereby showing that  $T - I \subseteq (q_D q_E q_1 D, q_D q_E q_3 E)$ . Since  $x \in T$  and  $x + q_D q_E n \in T$ ,  $x = q_D D k_D + q_E E k_E$  and  $x + q_D q_E n = q_D D c_D + q_E E c_E$  where  $k_D, c_D \in T_D$  and  $k_E, c_E \in T_E$ . But,  $q_D D k_D + q_E E k_E + q_D q_E n = x + q_D q_E n = q_D D c_D + q_E E c_E$ . If  $k_D = c_D$ , then  $q_D q_E n = q_E E (c_E - k_E)$ , which contradicts Lemma 3.8. Thus,  $k_D \neq c_D$ . We similarly see that  $k_E \neq c_E$ . Now,  $q_D q_E n = q_D D (c_D - k_D) + q_E E (c_E - k_E)$ . Since  $q_D q_E n > 0$ , we cannot have both  $k_D \geq c_D$  and  $k_E \geq c_E$ . By Lemma 3.8, we also cannot have both  $k_D \leq c_D$  and  $k_E \leq c_E$ . There are two cases to consider.

Case 1:  $k_D < c_D$  and  $k_E > c_E$

Case 2:  $k_E < c_E$  and  $k_D > c_D$

Suppose  $k_D < c_D$  and  $k_E > c_E$ . Now,

$$\begin{aligned} q_D D k_D + q_E E k_E + q_D q_E n &= q_D D c_D + q_E E c_E \\ \Leftrightarrow q_E E (k_E - c_E) + q_D q_E n &= q_D D (c_D - k_D) \\ \Leftrightarrow q_E E (k_E - c_E) + q_D q_E (q_2 E - q_1 D) &= q_D D (c_D - k_D) \\ \Leftrightarrow q_E E (k_E - c_E + q_D q_2) &= q_D D (c_D - k_D + q_E q_1) \\ \Rightarrow (k_E - c_E + q_D q_2) &= b q_D D \text{ for some } b \in \mathbb{N}. \end{aligned}$$

Since  $k_E > c_E$ ,  $k_E - c_E + q_D q_2 > 0$  and hence  $b \geq 1$ . We therefore have

$$\begin{aligned} (k_E - c_E + q_D q_2) &= b q_D D \\ \Leftrightarrow (k_E - c_E + q_D q_2) &= b q_D (q_2 + q_3) \\ \Leftrightarrow (k_E - c_E + q_D q_2) q_E E &= b q_D q_E E q_2 + b q_D q_E E q_3 \\ \Leftrightarrow (k_E - c_E) q_E E &= (b - 1) q_D q_E E q_2 + b q_D q_E E q_3 \\ \Leftrightarrow (k_E - c_E) q_E E &= q_D q_E E q_3 + (b - 1) q_D q_E E q_2 + (b - 1) q_D q_E E q_3. \end{aligned}$$

Now,  $(b - 1) q_D q_E E q_2 + (b - 1) q_D q_E E q_3 \in T$  since  $q_E E q_2, q_E E q_3 \in T$ . Since  $q_D D k_D, q_E E c_E$  are also in  $T$ , we have

$$\begin{aligned}
& (k_E - c_E)q_E E \in q_D q_E q_3 E + T \\
\Rightarrow & q_D D k_D + (k_E - c_E)q_E E + q_E E c_E \in q_D q_E q_3 E + T \\
\Rightarrow & q_D D k_D + q_E E k_E \in q_D q_E q_3 E + T \\
\Rightarrow & x \in q_D q_E q_3 E + T \subseteq (q_D q_E q_1 D + T) \cup (q_D q_E q_3 E + T).
\end{aligned}$$

Analogously, in Case 2 we see that  $x \in q_D q_E q_1 D + T \subseteq (q_D q_E q_1 D + T) \cup (q_D q_E q_3 E + T)$ . We have shown that  $T - I = (q_D q_E q_1 D, q_D q_E q_3 E)$ .

Since  $I = (0, q_D q_E n)$ ,

$$\begin{aligned}
& \{q_D q_E q_1 D, q_D q_E q_1 D + q_D q_E n, q_D q_E q_3 E, q_D q_E q_3 E + q_D q_E n\} \\
= & \{q_D q_E q_1 D, q_D q_E q_2 E, q_D q_E q_3 E, q_D q_E q_4 D\}
\end{aligned}$$

is a generating set for  $I + (T - I)$ . To complete the proof, we need to show that this generating set is minimal. Since  $q_D q_E q_1 D$  is the smallest element, it is clearly part of the minimal generating set. As a consequence of Lemma 3.8, neither  $q_D q_E q_2 E - q_D q_E q_1 D$  nor  $q_D q_E q_3 E - q_D q_E q_1 D$  are in  $T$ . Therefore,  $q_D q_E q_2 E \notin q_D q_E q_1 D + T$  and  $q_D q_E q_3 E \notin q_D q_E q_1 D + T$ . Since  $q_3 < D$ ,  $q_D q_3 < q_D D$  and we know from Remark 3.7 that every representation of  $q_D q_E q_3 E$  in  $T$  is a linear combination of the generators in  $G_E$ . However, this means that  $q_D q_E q_3 E \notin q_D q_E q_2 E + T$  since  $q_D(q_3 - q_2) \notin T_E$  by hypothesis. Thus,  $q_D q_E q_2 E$  and  $q_D q_E q_3 E$  must both be in the minimal generating set for  $T - I$ . Finally, we similarly know from Remark 3.7 that every representation of  $q_D q_E q_4 D$  in  $T$  is a linear combination of the generators in  $G_D$  so that  $q_D q_E q_4 D \notin q_D q_E q_2 E + T$  and  $q_D q_E q_4 D \notin q_D q_E q_3 E + T$ . But, this also means that  $q_D q_E q_4 D \notin q_D q_E q_1 D + T$  since  $q_E(q_4 - q_1) \notin T_D$ . Therefore,  $I + (T - I) = (q_D q_E q_1 D, q_D q_E q_2 E, q_D q_E q_3 E, q_D q_E q_4 D)$  and  $T$  is a  $2 \times 2$  brick. By definition,  $T_I = S$ .

**Example 3.9.** Let  $S = \langle 44, 45, 54, 55 \rangle$ . Then  $S$  is a unitary numerical semigroup with  $D = 11$ ,  $E = 9$ ,  $q_1 = 4$ ,  $q_2 = 5$ ,  $q_3 = 6$ ,  $q_4 = 5$ , and  $n = 1$ . Let  $T_D = \langle 4, 5 \rangle$ ,  $T_E = \langle 5, 6, 7, 8, 9 \rangle$ , and  $q_D = q_E = 1$ . It is clear that  $q_2, q_3 \in T_E$  and  $1 = q_D(q_3 - q_2) \notin T_E$ . It is also clear that  $q_1, q_4 \in T_D$  and  $q_E(q_4 - q_1) = 1 \notin T_D$ . Theorem 3.2 therefore guarantees that  $(T, I)$  is a  $2 \times 2$  brick, where  $T = \langle 4q_D D, 5q_D D, 5q_E E, 6q_E E, 7q_E E, 8q_E E, 9q_E E \rangle = \langle 44, 45, 54, 55, 63, 72, 81 \rangle$  and  $I = (0, 1)$ .

The following example shows that it is possible for the same  $T$  to produce  $2 \times 2$  bricks for two different unitary numerical semigroups.

**Example 3.10.** Let  $S = \langle 161, 180, 240, 259 \rangle$ . Then  $S$  is a unitary numerical semigroup with  $D = 7$ ,  $E = 60$ ,  $q_1 = 23$ ,  $q_2 = 3$ ,  $q_3 = 4$ ,  $q_4 = 37$ , and



$n = 19$ . Let  $T_D = \langle 3, 10 \rangle$ ,  $T_E = \langle 3, 4 \rangle$ , and  $q_D = q_E = 1$ . It is clear that  $q_2, q_3 \in T_E$  and  $1 = q_D(q_3 - q_2) \notin T_E$ . It is also clear that  $q_E(q_4 - q_1) = 14 \notin T_D$ . Since  $q_1 = 23 = 1 \cdot 3 + 2 \cdot 10 \in T_D$  and  $q_4 = 37 = 9 \cdot 3 + 1 \cdot 10 \in T_D$ , Theorem 3.2 guarantees that  $(T, I)$  is a  $2 \times 2$  brick, where  $T = \langle 3q_DD, 10q_DD, 3q_EE, 4q_EE \rangle = \langle 21, 70, 180, 240 \rangle$  and  $I = (0, 19)$ . That is,  $T_I = S$ .

Now, let  $U = \langle 180, 203, 217, 240 \rangle$ . Here,  $U$  is a unitary numerical semigroup with  $D = 60$ ,  $E = 7$ ,  $q_1 = 3$ ,  $q_2 = 29$ ,  $q_3 = 31$ ,  $q_4 = 4$ , and  $n = 23$ . Letting  $T_D = \langle 3, 4 \rangle$ ,  $T_E = \langle 3, 10 \rangle$ , and  $q_D = q_E = 1$ , it is clear that  $q_1, q_4 \in T_D$  and  $1 = q_E(q_4 - q_1) \notin T_D$ . It is also clear that  $q_D(q_3 - q_2) = 2 \notin T_E$ . Since  $q_2 = 29 = 3 \cdot 3 + 2 \cdot 10 \in T_E$  and  $q_3 = 31 = 7 \cdot 3 + 1 \cdot 10 \in T_E$ , Theorem 3.2 guarantees that  $(T, J)$  is a  $2 \times 2$  brick, where  $T = \langle 3q_DD, 4q_DD, 3q_EE, 10q_EE \rangle = \langle 21, 70, 180, 240 \rangle$  and  $J = (0, 23)$ . That is,  $T_J = U$ .

The following result is a rephrasing of part of Proposition 10 in [4].

**Theorem 3.11.** Let  $T_D = \langle s_1, s_2, \dots, s_m \rangle$  and  $T_E = \langle r_1, r_2, \dots, r_k \rangle$  be numerical semigroups. Let  $P$  and  $Q$  be relatively prime positive integers such that  $P \in T_E$  and  $Q \in T_D$ . Then the numerical semigroup  $T = PT_D + QT_E$  satisfies the following properties:

- (1) The conductor of  $T$  is  $c_1P + c_2Q + (P - 1)(Q - 1)$ , where  $c_1$  and  $c_2$  are the conductors of  $T_D$  and  $T_E$ , respectively, and
- (2)  $T$  is symmetric if and only if  $T_D$  and  $T_E$  are symmetric.

**Corollary 3.12.** Let  $T$  be a numerical semigroup as defined in Theorem 3.2. Then  $g(T) = g(T_D)q_DD + g(T_E)q_EE + q_DDq_EE$  and  $T$  is symmetric if and only if both  $T_D$  and  $T_E$  are symmetric.

**Proof.** Letting  $P = q_DD$  and  $Q = q_EE$ , we see that the hypotheses of Theorem 3.11 are satisfied. Thus,

$$\begin{aligned} g(T) + 1 &= (g(T_D) + 1)q_DD + (g(T_E) + 1)q_EE + (q_DD - 1)(q_EE - 1) \\ &= g(T_D)q_DD + g(T_E)q_EE + q_DDq_EE + 1 \end{aligned}$$

and the result follows.

We note that the formula in Corollary 3.12 agrees with the one in [10] when  $T$  is unitary, that is, when  $T = S$ . In this case, we have  $q_D = q_E = 1$ ,



$T_D = \langle q_1, q_4 \rangle$ , and  $T_E = \langle q_2, q_3 \rangle$ , so that  $g(T_D) = q_1q_4 - q_1 - q_4$  and  $g(T_E) = q_2q_3 - q_2 - q_3$ . By Lemma 2.4 in [10], we have

$$\begin{aligned}
g(T) &= (q_3 - 1)a_2 + (q_1 - 1)a_4 + n \\
&= (q_3 - 1)q_2E + (q_1 - 1)q_4D + (q_2E - q_1D) \\
&= q_2q_3E + (q_1q_4 - q_1 - q_4)D \\
&= (q_1q_4 - q_1 - q_4)D + (q_2q_3 - q_2 - q_3)E + (q_2 + q_3)E \\
&= g(T_D)D + g(T_E)E + DE \\
&= g(T_D)q_DD + g(T_E)q_EE + q_DDq_EE.
\end{aligned}$$

**Observations 3.13.** Consider the case when the minimal generating set of  $T$  contains exactly four elements. By Corollary 3.12,  $T$  is symmetric. To see this, we observe that since  $q_1, q_4 \in T_D$  and  $q_4 - q_1 \notin T_D$ , the minimal generating set for  $T_D$  must contain at least two elements. The minimal generating set for  $T_E$  must similarly contain at least two elements. Since the minimal generating set for  $T$  has exactly four elements, the minimal generating sets for  $T_D$  and  $T_E$  must have precisely two elements each. It is well known that numerical semigroups with exactly two elements are symmetric. Therefore,  $T_D$  and  $T_E$  are symmetric and hence  $T$  is symmetric. Moreover,  $T$  is a complete intersection since it is the gluing of two numerical semigroups of embedding dimension two (see Chapter 8 in [14]).

We note that since  $T_D$  and  $T_E$  are semigroups with fewer and smaller generators than  $T$ , it is much faster and easier to verify that they are symmetric than attempting to verify that  $T$  is symmetric directly. We demonstrate the connection between Theorem 3.2, Corollary 3.12, and torsion in tensor products with some examples.

**Example 3.14.** Let  $S = \langle 36, 44, 55, 63 \rangle$ . Then  $S$  is a unitary numerical semigroup with  $D = 9$ ,  $E = 11$ ,  $q_1 = 4$ ,  $q_2 = 4$ ,  $q_3 = 5$ ,  $q_4 = 7$ , and  $n = 8$ .

Let  $T_D = \langle 4, 6, 7 \rangle$ ,  $T_E = \langle 4, 5, 6 \rangle$ , and  $q_D = q_E = 1$ . It is clear that  $q_2, q_3 \in T_E$  and  $1 = q_D(q_3 - q_2) \notin T_E$ . It is also clear that  $q_1, q_4 \in T_D$  and  $q_E(q_4 - q_1) = 3 \notin T_D$ . By Theorem 3.2,  $(T, I)$  is a  $2 \times 2$  brick, where  $T = \langle 36, 44, 54, 55, 63, 66 \rangle$ ,  $I = \langle 0, 8 \rangle$ , and  $T - I = \langle 36, 55 \rangle$ .

If  $k$  is a field, we have the monomial domain  $R = k[[t^{36}, t^{44}, t^{54}, t^{55}, t^{63}, t^{66}]]$  with monomial fractional ideals  $J = (1, t^8)$  and  $J^{-1} = (t^{36}, t^{55})$ . If we let  $K$  be the quotient field of  $R$  and  $\nu : K^* \rightarrow \mathbf{Z}$  be the standard valuation mapping, then we see that  $T = \nu(R)$ . For  $J \otimes_R J^{-1}$  to be torsion free, we must have  $\mu_R(J) + \mu_R(J^{-1}) = \mu_R(JJ^{-1})$  and hence  $\mu_T(\nu(J)) + \mu_T(T - \nu(J)) = \mu_T(\nu(JJ^{-1}))$ . This holds since  $(T, I)$  is a  $2 \times 2$  brick.

However, it is easy to verify that  $T_D$  and  $T_E$  are both symmetric. By Corollary 3.12,  $T$  is also symmetric. We therefore know that  $J \otimes_R J^{-1}$  has non-zero torsion.

Now, let  $T_D = \langle 4, 7, 10 \rangle$  and  $T_E = \langle 4, 5, 11 \rangle$ . It is clear that  $q_2, q_3 \in T_E$  and  $1 = q_D(q_3 - q_2) \notin T_E$ . It is also clear that  $q_1, q_4 \in T_D$  and  $q_E(q_4 - q_1) = 3 \notin T_D$ . By Theorem 3.2,  $(T, I)$  is a  $2 \times 2$  brick, where  $T = \langle 36, 44, 55, 63, 90, 121 \rangle$ ,  $I = (0, 8)$ , and  $T - I = (36, 55)$ .

Let  $R = k[[t^{36}, t^{44}, t^{55}, t^{63}, t^{90}, t^{121}]]$  with monomial fractional ideals  $J = (1, t^8)$  and  $J^{-1} = (t^{36}, t^{55})$ . Since  $(T, I)$  is a  $2 \times 2$  brick, we know that  $\mu_R(J) + \mu_R(J^{-1}) = \mu_R(JJ^{-1})$ . It is easy to verify that  $T_D$  is symmetric, while  $T_E$  is not. By Corollary 3.12, we know that  $T$  is not symmetric. Therefore, we cannot rule out the possibility that  $J \otimes_R J^{-1}$  is torsion free. This will have to be determined through other established methods.

### Open Questions.

- (1) Does the converse of Theorem 3.2 hold? That is, if  $(T, I)$  is a  $2 \times 2$  brick such that the minimal generating set of  $I + (T - I)$  is a constant multiple of a unitary set, must it be of the form described?
- (2) If  $(T, I)$  is a  $2 \times 2$  brick, is the minimal generating set of  $I + (T - I)$  always a constant multiple of a unitary set?
- (3) We have recently shown that there are perfect bricks of every dimension. That is, for any  $k, m \in \mathbf{Z}$  such that  $k, m \geq 2$ , there is a perfect  $k \times m$  brick. Can we construct families of  $k \times m$  bricks using ideas analogous to those used in Theorem 3.2?

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